THERMAL COSMIC RADIATION AND THE FORMATION OF PROTOGALAXIES

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Friedmann[1] first discussed the dynamic solution of Einstein's relativistic equation for the gravitational field in the case of a homogeneous and isotropic distribution of mass. His result for the time rate of change of the cosmic scale l (the proper distance between arbitrary points) may be written without the cosmological term for an expanding space as[2]

$$\frac{1}{l}\frac{dl}{dt} = \sqrt{\frac{8\pi G}{3}\rho - \frac{\alpha^2 c^2}{l^2}}, \tag{1}$$

where G is Newton's gravitational constant, c the velocity of light, ρ the mean total density in mass units, and α is a constant equal to the ratio l/R, where R is the radius of curvature of space-time.

If we consider space to be filled with a mixture of noninterconverting blackbody radiation and matter in the form of baryons (subject to conservation of mass) in thermodynamic equilibrium, then the mean total density may be written as

$$\rho = \rho_r + \rho_m = \frac{aT^4}{c^2} + \frac{b}{l^3}, \tag{2}$$

where a is the radiation density constant and b is a constant whose value depends on the measure of the cosmic scale. It is convenient to choose for l the side of a cube containing 1 gm of matter. If the matter density in the universe at present is 7×10^{-31} gm/cm^3, then the present value of the cosmic scale factor is 1.13×10^{10} cm, and the quantity $L = l/l_0$ changes from 0 at the start of the universal expansion to 1 at present. It should be noted in what follows that we ignore the possible role of neutrinos throughout the expansion and other elementary particles in the early stages.

The dependence of ρ_r on l can be obtained from the energy equation for the cosmological model with the requirement of conservation of mass,[3] but is perhaps more simply obtained by invoking Wien's law. According to this, for blackbody radiation $\lambda_{\max}T$ = const.; moreover, in the universal expansion all lengths, including wavelengths, should change in proportion to the cosmic scale. Therefore: $T \propto l^{-1}$ and $aT^4c^{-2} \propto l^{-4}$. Implicit in the statement that we are dealing with the adiabatic expansion of a uniformly expanding space containing a homogeneous, isotropic distribution of mass is the requirement that all material bodies (be they atoms, stars, or galaxies) participate in the expansion as though they were particles of a monatomic gas,[4] with an adiabatic exponent $^5/_3$. It then follows that for matter $\rho_m^{2/3} \propto T \propto l^{-2}$, and so there must have been a transfer of energy from the blackbody radiation to matter in order that temperatures be equalized. One can show, however, that the net effect of this transfer on the radiation temperature is indeed quite small. We recall that the heat capacities of radiation and of the matter present are given by

$$C_r = 4aT^3 \quad \text{ergs/cm}^3\ {}^\circ\text{K} \qquad C_m = \frac{3}{2}nk \quad \text{ergs/cm}^3\ {}^\circ\text{K}, \tag{3}$$

where n is the number density of particles. If we take the present radiation temperature to be 3°K, and the present number density to be $(7 \times 10^{-31}$ gm/cm$^3)/(1.67 \times 10^{-24}$ gm/particle$) = 4.2 \times 10^{-7}$/cm^3, then $C_r(\text{now}) = 8.2 \times 10^{-13}$ ergs/cm^3 and $C_m(\text{now}) = 8.7 \times 10^{-23}$ ergs/cm^3. Moreover, since $C_r \propto 1/l^3 (T \propto 1/l)$ and $C_m \propto 1/l^3 (n \propto 1/l^3)$, the ratio $C_r/C_m = 8aT^3/(3nk) \cong 10^{10}$ was the same constant through the entire evolution of the universe.[5] As will be seen later, in this cosmological model there would have been a decrease in the density of matter from an initial dense state at, say, 1 sec to the present time of a factor of $\sim 10^{-30}$. In the absence of radiation and if the expansion were adiabatic, then the drop in temperature would have been by a factor $(10^{-30})^{2/3} = 10^{-20}$. If the temperature at $t = 1$ sec had been 10^{11} °K, then the present temperature of matter would have been $\sim 10^{-9}$ °K. If matter is now in fact in equilibrium with 3°K blackbody radiation, there would have to have been a net heat transfer from radiation of the order of $(3°\text{K}-10^{-9}\ °\text{K}) \times 8.7 \times 10^{-23} = 2.6 \times 10^{-22}$ ergs/cm^3. The temperature change in the radiation field required to supply this energy is then $2.6 \times 10^{-22}/8.2 \times 10^{-13} \cong 0.34 \times 10^{-9}$ °K. It may be noted that for this cosmological model in which matter is conserved, the total energy is not;[3] the energy in a given volume which is not accounted for represents the pdV work done in the adiabatic expansion. It is readily calculable[3] but not readily accountable in terms of whence it goes, since the calculation proceeds as though the matter-radiation mix were doing work on a container in the expansion—a container of somewhat dubious reality.

Early studies of the physical conditions required for nucleosynthesis in the initial stages of an expanding universe[6,7] indicated that one had then radiation density very much larger than the density of matter and completely dominating the behavior of the cosmological model. With $\rho_r \gg \rho_m$ and at early times, equation (1) can be written

$$\frac{1}{l}\frac{dl}{dt} = \sqrt{\frac{8\pi G}{3}\frac{aT^4}{c^2}}, \tag{4}$$

or, using $l \propto 1/T$

$$-\frac{1}{T^3}\frac{dT}{dt} = \sqrt{\frac{8\pi Ga}{3c^2}}, \tag{5}$$

which can be integrated as

$$T = \left(\frac{3c^2}{32\pi Ga}\right)^{1/4} \frac{1}{\sqrt{t}} = \frac{1.52 \times 10^{10}}{\sqrt{t}}\ {}^\circ\text{K}, \tag{6}$$

where we have chosen the constant of integration so that $T \rightarrow \infty$ as $t \rightarrow 0$. It follows that the density of radiation can be written as $\rho_r = 4.42 \times 10^5 t^{-2}$ gm/cm^3. The density of matter varies as T^3, so that during the radiation-controlled expansion $\rho_m = \rho_0 t^{-3/2}$, where ρ_0 is a constant whose value must be determined by additional information. As will be discussed shortly, recent observations of cosmic blackbody radiation now make it possible to fix ρ_0 from present observables. Prior to this, one could evaluate ρ_0 only by recourse to the description of nucleosynthesis processes in the early stages of the universal expansion.[9] In the determination of physical conditions required to effect agreement between calculated and observed cosmic element abundances, the constant ρ_0 played the role of the only free parameter in the calculations, with temperature and radiation density being prescribed by the radiation-controlled cosmological model.

It is of some interest to restate the several early attempts to evaluate ρ_0, since having ρ_0 and ρ_r prescribed made it possible to do the inverse calculation of predicting the present cosmic blackbody radiation. The early calculation of cosmic element abundances according to a successive neutron capture theory[10] led to a value $\rho_0 = 1.5 \times 10^{-3}$ gm cm^{-3} sec$^{3/2}$. Consideration of the formation of only deuterons from neutrons and protons,[11] requiring the residual hydrogen to have an abundance of $\sim$50 per cent, led to a value $\rho_0 = 4.8 \times 10^{-4}$ gm cm^{-3} sec$^{3/2}$. Another calculation involving graphical solutions of equations (1) and (2), rather than nucleosynthesis evidence,[12] led to a value for $\rho_0 \cong 1 \times 10^{-2}$ gm cm^{-3} sec$^{3/2}$.

Given these values of ρ_0 and the present mean density of matter in the universe,[13] it is a relatively simple matter to calculate the present value for cosmic blackbody radiation. It follows from the assumption of an adiabatic expansion in which $\rho_r l^4$ = constant and $\rho_m l^3$ = constant that so long as there is no matter-radiation interconversion, one must have throughout the expansion[8]

$$\rho_r \rho_m{}^{-4/3} = \text{const.} = \rho_{r''} \rho_{m''}{}^{-4/3}, \qquad (7)$$

where ρ_r and ρ_m are values at *any* time and $\rho_{r''}$ and $\rho_{m''}$ are present values. By using equation (7) with values of ρ_0 and $\rho_{m''}$ then current, Alpher and Herman in 1948[11] and 1949[8] calculated a value of $\rho_{r''}$ equivalent to $\sim$5°K for the present universal blackbody temperature. It was noted that the energy densities of $\sim$5°K blackbody radiation and the energy density of starlight would be similar (not so the spectral energy distributions, of course). On the basis of a revised estimate of the present universal density of matter due to Behr, since discredited, Alpher and Herman proposed in 1951[10] that a value of as much as 28°K might be expected for the residual radiation. Finally, in 1953, Gamow,[12] using an approximate analysis which does not require a knowledge of ρ_0, obtained an estimate of 7°K for the present residual radiation.

The situation is now reversed. Recent observations demonstrate a highly isotropic thermal cosmic radiation corresponding to a blackbody at 3°K.[14] There seems little doubt that this represents the red-shifted thermal radiation associated with a cosmological model of the form discussed here; at least no satisfactory alternative explanation has been put forth. Thus it is now possible to state the value of ρ_0 unambiguously in terms of present observables ($\rho_{m''}$ and $\rho_{r''}$) without resort to a calculation of primordial nucleosynthesis. Taking ρ_r at $t = 1$ sec to be 4.42×10^5, fixed purely by the cosmological model, one then has $\rho_0 = 1.28 \times 10^{29} \times \rho_{m''}$, which has the value 0.090 for $\rho_{m''} = 7 \times 10^{-31}$ gm/cm^3, and 0.38 for $\rho_{m''} = 3 \times 10^{-30}$ gm/cm^3.

The fact that $\rho_{r''}$ is now an observable makes it possible to define more precisely some of the features of the homogeneous, isotropic nonstatic cosmological model.

In particular, a calculation of the age of this model becomes more meaningful. It will be recalled that some past objections and in fact some alternative cosmological theories have resulted from the fact that the model discussed here gave too short an age for the universe. Equations (1) and (2) can be integrated analytically to yield both the present age and also the behavior of the several variables with time during the expansion. The solutions were discussed generally by Heckmann[15] and more recently by Alpher and Herman.[8] It is convenient to combine equations (1) and (3) in the following form, anticipating an imaginary radius of curvature

$$\frac{1}{L}\frac{dL}{dt} = \sqrt{\gamma\left(\frac{\rho_{m''}}{L^3} + \frac{\rho_{r''}}{L^4}\right) + \frac{K_2}{L^2}}, \tag{8}$$

where $\gamma = 8\pi G/3$, $L = l/l_0$, with l_0 the cosmic scale chosen so that, as described earlier, it is the side of a cube now containing a gram of matter plus radiation, and $L = 1$ refers to now, $\rho_{m''}$ and $\rho_{r''}$ are as before the present values, and $K_2 = c^2/|R_0|^2$ with R_0 the present value of the radius of curvature. Equation (8) integrates to yield

$$t = K_1 + K_L/K_2 - (\gamma\rho_{m''}/2K_2^{3/2}) \ln\,[K_L + K_2^{1/2}L + \gamma\rho_{m''}/(2K_2^{1/2})] \text{ sec} \tag{9}$$

where

$$K_1 = (\gamma\rho_{m''}/2K_2^{3/2}) \ln\,[(\gamma\rho_{r''})^{1/2} + (\gamma\rho_{m''}/2K_2^{1/2})] - (\gamma\rho_{r''}/K_2^2)^{1/2} \tag{10}$$

and

$$K_L = (\gamma\rho_{r''} + \gamma\rho_{m''}L + K_2L^2)^{1/2}. \tag{11}$$

The constant K_2 can be evaluated by taking $L = 1$ now and noting that $(1/L) \times (dL/dt)_{L=1}$ is the present value of Hubble's expansion rate parameter, H_0. Thus

$$K_2 = H_0^2 - \gamma(\rho_{m''} + \rho_{r''}). \tag{12}$$

The present age of the universe according to this model follows from equation (9) with $L = 1$. Table 1 shows the age of the universe (in eons = 10^9 years) calculated for three different values of H_0 and for a selection of values of the present density of matter and blackbody radiation temperature. Attention is called to several points in this table. First, the age associated with presently used values of $\rho_{m''} = 7 \times 10^{-31}$ gm/cm^3 and $T_{BB} = 3$°K is about 9.3 eons, which is in satisfactory agreement with present estimates of the age of the universe

TABLE 1

AGE OF UNIVERSAL EXPANSION (EONS)

H_0 = 75 km/sec/Mpc = 0.243×10^{-17} sec^{-1}

$\rho_{m''}$*	T_{BB}°K 1	3	5	7	9
0.1	12.8	12.7	12.6	12.4	12.1
0.5	12.2	12.2	12.1	12.0	11.8
1.0	11.8	11.7	11.7	11.6	11.4
5.0	9.91	9.91	9.90	9.87	9.81
10.0	8.79	8.79	8.78	8.77	8.74

H_0 = 100 km/sec/Mpc = 0.324×10^{-17} sec^{-1}

$\rho_{m''}$*	T_{BB}°K 1	3	5	7	9
0.1	9.65	9.63	9.55	9.41	9.23
0.5	9.37	9.36	9.31	9.21	9.07
1.0	9.11	9.11	9.08	9.01	8.89
5.0	8.02	8.02	8.01	7.99	7.95
10.0	7.30	7.30	7.29	7.28	7.26

H_0 = 125 km/sec/Mpc = 0.405×10^{-17} sec^{-1}

$\rho_{m''}$*	T_{BB}°K 1	3	5	7	9
0.1	7.75	7.73	7.68	7.59	7.47
0.5	7.58	7.58	7.54	7.48	7.38
1.0	7.43	7.43	7.40	7.35	7.27
5.0	6.73	6.72	6.72	7.60	6.67
10.0	6.23	6.22	6.22	6.21	6.19

* In 10^{-30} gm/cm^3.

based on other considerations. Second, the calculated ages are relatively insensitive to the choice of the present cosmic blackbody radiation temperature, which explains why previous estimates of the blackbody radiation temperature were comparatively close to the observed value despite grossly different values for H_0 or the age chosen for the universe.

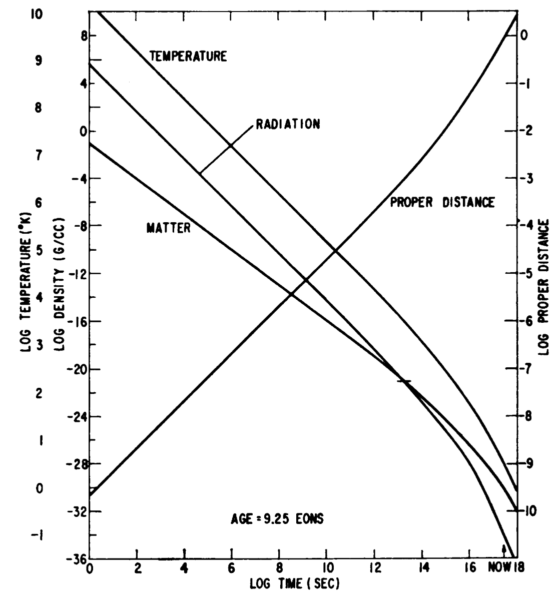

FIG. 1.—Variation of dimensionless proper distance, temperature, matter, and radiation densities in a nonstatic, homogeneous, isotropic, cosmological model of noninterconverting matter and radiation for H_0 = 100 km/sec/Mpc; $\rho_{m''}$ = 7 × 10^{-31} gm/cm^3, and T_{BB} = 3°K.

Using equation (9), we have calculated the variation of the dimensionless scale L, temperature T, radiation density ρ_r, and matter density ρ_m as a function of epoch for $T_{BB} = 3$°K, $\rho_{m''} = 7 \times 10^{-31}$ gm/cm^3, and H_0 = 100 km/sec/Mpc. The results are plotted in Figure 1. It shows a transition from a radiation-controlled expansion to a matter-controlled expansion at $t \cong 10^6$ years after the start of the expansion.[12a] It may be noted that a value of $\rho_{m''} \cong 2 \times 10^{-31}$ gm/cm^3 was suggested recently by Wagoner, Fowler, and Hoyle[13] as consistent with the early high densities required for primordial nucleosynthesis, as well as with the presently observed $T_{BB} = 3$°K.

The fact that the present blackbody radiation is now an observable bears directly on several problems of cosmological interest, namely primordial nucleosynthetic processes and the problem of the origin of protogalaxies. We do not propose to discuss the former subject here since it was recently the subject of a critical review.[13] In early discussions on the subject of protogalaxies[16] the suggestion was made that the formation of condensations in an expanding universe should be subject to two conditions. First, the radius R of the rudimentary condensation should have been sufficiently small that the recession velocity with respect to the center of particles located at the surface would have been smaller than the mean velocity v of their random motion. This condition can be written as

$$R(t) \times H(t) < v(t) \quad \text{or} \quad R_{\max} = v(t)/H(t), \tag{13}$$

where $H(t)$ is Hubble's parameter at time t. When this condition was proposed, Gamow and Teller supposed the "particles" in the protogalaxy to be stars, whereas it is of course now known that with this assumption one cannot account for the shapes of elliptic galaxies—mean free paths of stars in such galaxies are much greater than galactic dimensions.[17] Now the particles are supposed to be primordial, primarily hydrogen, with the velocity distribution determined from the universal temperature.

As a second condition one supposes that the rudimentary condensation should have been sufficiently large that the escape velocity of particles from the surface would have exceeded the mean velocity of their thermal motion. This is equivalent to the familiar Jeans criterion for gravitational instability in a static medium,[18] and can be written

$$\frac{4\pi G}{3}\rho(t)R^2(t) > \frac{1}{2}v^2(t) \quad \text{or} \quad R_{\min} = \sqrt{\frac{3}{8\pi G}\frac{v^2(t)}{\rho}}. \tag{14}$$

(Note that this differs by less than a factor of 10 from the classical instability criterion of Jeans,[18] which can be written $D_{\min} = \sqrt{5\pi v^2/(9Gm_H\rho\bar{\mu})}$, where $\bar{\mu}$ is the mean molecular weight of the medium.) This seemingly sensible pair of conditions for the size of a condensation is not internally consistent, for if we combine equations (13) and (14) and replace $H(t) = (dL/dt)/L$ from equation (8), then

$$R_{\max}/R_{\min} = \{1 + 3K_2/[8\pi G\rho(t)L^2(t)]\}^{-\frac{1}{2}}, \tag{15}$$

which is less than unity for all t, and is independent of $v(t)$. Thus it does not resolve the inconsistency if one uses some other measure of particle speed so long as the same measure is used in equation (13) and (14). Clearly a more sophisticated approach is required to the question of the size and mass of condensations from the expanding medium.

It is also disappointing that neither equation (13) nor equation (14) separately gives reasonable values for the size of a condensation when applied to that time in the universal evolution when $\rho_m \cong \rho_r$. For $T_{\text{now}} = 3°\text{K}$, $\rho_{m''} = 7 \times 10^{-31}$ gm/cm^3, and at the time when $\rho_m = \rho_r$, one finds $\rho = 0.76 \times 10^{-21}$ gm/cm^3, $T = 3090°\text{K}$, and $R_{\max} \lesssim R_{\min} \cong 44$ light years, $M/M_\odot \simeq 1.2 \times 10^5$, which are far different from values required for a protogalaxy.

A rather different result is obtained if one considers the classical Jeans criterion[18] for the characteristic size of a condensation, $\lambda^2 = \pi u^2(t)/(G\rho)$, where $u(t)$ is the velocity of sound in the medium, where, in contradistinction to the conventional usage, ρ is the density when $\rho_m = \rho_r$, and $u(t)$ is taken for an equilibrium mixture of noninterconverting matter and radiation. For the conditions in which we are interested, and with ρ_r and ρ_m related by equation (7), $u(t)$ can be calculated from[19]

$$u^2(t) = \frac{5kT}{3m_H}\left\{1 + \frac{32a^2T^6}{9k^2}\left[\frac{5\rho_m}{m_H}\left(\frac{\rho_m}{m_H} + \frac{8aT^3}{3k}\right)\right]^{-1}\right\} \cong \frac{4}{9}c^2\frac{\rho_r}{\rho_m} \tag{16}$$

where c is the velocity of light. Thus at $\rho_r = \rho_m$, $u(t) = 2c/3 = 2 \times 10^{10}$ cm/sec, and one obtains for the condensation $\lambda \cong 5.3 \times 10^6$ light years, $M/M_\odot \cong 10^{20}$. Not only are these not reasonable numbers in terms of the properties of galactic clusters, but λ exceeds the horizon imposed by causality at the time when $\rho_m = \rho_r$ for these conditions (see Fig. 1).

A related but perhaps more meaningful criterion for a condensation might be obtained as follows. On the one hand, there should be a characteristic time for the formation of a fluctuation, whose value would be of the order of the transit time of a sound wave across the dimension of the fluctuation (diameter/sound speed). On the other hand, there should be a characteristic time associated with the universal expansion, whose value would be of the order of the reciprocal of the Hubble parameter at the time of the fluctuation. For a condensation to overcome the expansion then, we could require $R(t)/u(t) < H(t)/2$. If we restrict ourselves to values of ρ_r/ρ_m such that $u(t) < c$, (i.e., $\rho_r < 9\rho_m/4$) then equation (8) yields

$$R(t) \leqq \frac{c}{3}\sqrt{\frac{\rho_r}{\rho_m}}\left[\frac{8\pi G}{3}(\rho_r + \rho_m) + \frac{K_2}{L^2}\right]^{-\frac{1}{2}}. \tag{17}$$

Again for $T_{\text{now}} = 3°\text{K}$, $\rho_{m''} = 7 \times 10^{-31}$ gm/cm^3, and at $\rho_m = \rho_r$, this yields $R(t) \leq 3.6 \times 10^5$ light years and $M/M_{\odot} \leq 6.3 \times 10^{16}$. These are not unreasonable values for a protogalaxy, particularly if in subsequent evolution the protogalaxy expanded to a mean density of $\sim 10^{-24}$ with a consequent increase in $R(t)$ by a factor $(0.76 \times 10^{-21}/10^{-24})^{1/3} \cong 9$, where 0.76×10^{-21} gm/cm^3 was the density when $\rho_m \cong \rho_r$.

The question of how protogalaxies may have formed, or to put it more generally, of how structural differentiation occurred in the expanding universe, is clearly a very open one. There is considerable effort now being made to assess the role of the cosmic blackbody radiation, primordial magnetic fields, turbulence, and anisotropy in the differentiation process. We do not propose to review these efforts but merely to mention several. In particular, the papers of Peebles,[20] Nariai, Tomita, and Kato,[21] Thorne,[22] and Harrison[23] are representative of recent work directed to this end; in these papers conclusions range from finding it conceptually possible to develop a scheme for structural differentiation to finding that structural differentiation must have occurred so early in the expansion as to require the existence of structure as an initial cosmological condition.

Further consideration of the influence of the background radiation on protogalaxy formation and on the observed distribution of luminous matter is reserved for a future paper. It should be noted that none of the conclusions or numerical results described in this paper are significantly altered by a more careful description of the early stages of the expansion in which the interconversion of radiation and elementary particles is admitted.[24, 13]

Summary.—Some consequences of the recent observational verification of an evolutionary cosmology are examined. In particular, the observed 3°K cosmic blackbody radiation leads to a reasonable theoretical age of the universe and defines within narrow limits the universal density of matter at early times in the expansion. The question of the formation of protogalaxies remains a dilemma, although a new approach is suggested.

[1] Friedmann, A., *Z. Physik.*, **10**, 377 (1922); *ibid.*, **21**, 326 (1924). (One of us (G. G.), while a student at the University of Leningrad, took a course from Prof. Friedmann on the special and general theories of relativity.)

[2] Tolman, R. C., *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon Press, 1934), p. 396.

[3] *Ibid.*, p. 408.

[4] *Ibid.*, p. 422.

[5] There are other interesting interpretations of the numerical value of the ratio C_r/C_m which appears to be a new dimensionless cosmological quantity. For example, the ratio of the radiation pressure to pressure due to matter is $p_r/p_m = (^1/_3)aT^4/(nkT) = (^1/_8)C_r/C_m$; this suggests a major role for radiation in the structural evolution of the universe. Again, the dimensionless entropy per nucleon is $S/(nk) \cong S_r/(nk) = 4aT^3/(3nk) = (^1/_2)C_r/C_m$. Finally, and perhaps most interesting, we consider the number of photons per nucleon. This can be shown to be $n_{ph}/n \cong (^1/_7)\, C_r/C_m$.

In terms of observables $n_{ph}/n = 20.3\ T^3/n = 20.3 T_0^3 m_H/\rho_{m''}$. An interesting discussion of the last two points is given by Zeldovich, Y. B., *Sov. Phys. Usp.* (English transl.), **9**, 602 (1967).

[6] Alpher, R. A., thesis, George Washington University (1948).

[7] Gamow, G., *Phys. Rev.*, **74**, 505 (1948); Gamow, G., *Nature*, **162**, 680 (1948).

[8] Alpher, R. A., and R. Herman, *Phys. Rev.*, **75**, 1089 (1949).

[9] Alpher, R. A., H. A. Bethe, and G. Gamow, *Phys. Rev.*, **73**, 803 (1948). See also: Gamow, G., in *Perspectives in Modern Physics*—Essays in Honor of H. A. Bethe (New York: John Wiley & Sons, 1966), pp. 443–448; Alpher, R. A., *Phys. Rev.*, **74**, 1577 (1948); Alpher, R. A., and R. Herman, *Phys. Rev.*, **74**, 1737 (1948); Alpher, R. A., R. Herman, and G. Gamow, *Phys. Rev.*, **74**, 1198 (1948); *ibid.*, **75**, 701 (1949); *ibid.*, **75**, 332A (1949); Alpher, R. A., and R. Herman, *Phys. Rev.*, **75**, 1089 (1949); Fermi, E., and A. Turkevich, unpublished work described in review by Alpher, R. A., and R. Herman, *Rev. Mod. Phys.*, **22**, 153 (1950).

[10] Alpher, R. A., and R. Herman, *Phys. Rev.*, **84**, 60 (1951).

[11] Gamow, G., *Nature*, **162**, 680 (1948); Alpher, R. A., and R. Herman, *Nature*, **162**, 776 (1948).

[12] Gamow, G., *Det. Kong. Danske Vid. Sels.*, **27**, no. 10 (1953).

[12a] Note that the magnitude of the density when $\rho_m = \rho_r$ follows from equation (7) as $\rho_c = \rho_{m''}^4/\rho_{r''}^3$, where, since $\rho_{m''}/L_c^3 = \rho_{r''}/L_c^4$, one has $L_c = \rho_{r''}/\rho_{m''}$. Neither ρ_c nor L_c at this crossover depends on H_0. If one knows H_0, then the time at crossover follows from equation (9).

[13] Recent studies of nucleosynthesis in the early stages of an expanding universe give somewhat different and improved values of ρ_0, as, for example, in Wagoner, R. V., W. A. Fowler, and F. Hoyle, *Ap. J.*, **148**, 3 (1967) or Peebles, P. J. E., *Phys. Rev. Letters*, **16**, 410 (1966), Peebles, P. J. E., *Ap. J.*, **146**, 542 (1966). In these papers one finds instead of $\rho_m = \rho_0 t^{-3/2}$ the equivalent expression $\rho = hT_9^3$, whence $\rho_0 = (h/27)[3c^2/(32\pi aG)]^{3/4} = 3.47 \times 10^3 h$. Wagoner, Fowler, and Hoyle find h to give a best fit to observed abundances when $10^{-6} < h < 10^{-4}$ or $3.5 \times 10^{-3} < \rho_0 < 3.5 \times 10^{-1}$. On the basis of H/He abundance ratios only, Peebles finds $h = 3 \times 10^{-5}$ or $\rho_0 = 0.10$.

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